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Almost automorphic solutions for nonautonomous stochastic differential equations

Miaomiao Fu

Department of Mathematics, Changchun Normal College, Changchun 130032, People's Republic of China

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ABSTRACT

In this paper, the concept of distributional almost automorphy for stochastic processes is introduced. Under some dissipative conditions, we obtain the existence and uniqueness of distributionally almost automorphic solutions to nonautonomous stochastic equations on any real separable Hilbert space.

1. Introduction

The concept of almost automorphy was introduced by Bochner [1] in relation to some aspects of differential geometry. It is a generalization of almost periodicity. Almost automorphic functions are characterized by the following property: for a continuous function $f : \mathbb{R} \to \mathbb{R}^d$, for any sequence $\{s'_n\}$ of real numbers, we can extract a subsequence $\{s_n\}$ such that for some function g, the following:

$$g(t) = \lim_{n\to\infty} f(t+s_n) \quad \text{and} \quad \lim_{n\to\infty} g(t-s_n) = f(t),$$

hold for each $t \in \mathbb{R}$. The convergence is simply pointwise, while one requires uniform convergence for almost periodicity. Almost automorphy has been studied by Veech [2,3], Shen and Yi [4] for classical exposition.

The study of periodic, almost periodic or pseudo-almost periodic solutions for finite and infinite dimensional affine Itô equations can be found in [5–15]. Recently, the case of square-mean almost periodic stochastic differential equations was considered in [16], where a unique almost automorphic solution was shown under the exponential stability of the linear part.

In this paper, we introduce the concept of almost automorphic mapping with values in the space of probability measures, and obtain the existence and uniqueness of a solution with the property of being almost automorphic in the distributional sense to a class of nonautonomous semilinear stochastic differential equations.

The paper is organized as follows. In Section 2, we introduce the notion of distributionally almost automorphic processes and study some of their basic properties. In Section 3, we prove the existence and uniqueness of distributionally almost automorphic mild solutions to nonautonomous stochastic differential equations.

E-mail addresses: 653185949@qq.com, mmfucaathy@yahoo.com.cn.

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2. Processes that are almost automorphic in distribution

Assume that $\mathbb{H}$ is a real separable Hilbert space and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ denote a filtered probability space, and $\mathcal{L}_2(\mathbb{R}, \mathbb{H})$ stand for the space of all $\mathbb{H}$-valued random variables such that

$$\mathrm{E}\|x\|^2 = \int_\Omega \|x\|^2 d\mathrm{P} < \infty.$$

For $x \in \mathcal{L}_2(\mathrm{P}, \mathbb{H})$, choose its norm as follows:

$$\|x\|_2 := \left(\int_\Omega \|x\|^2 d\mathrm{P}\right)^{\frac{1}{2}}.$$

For a symmetric nonnegative operator $Q \in \mathcal{L}_2(\mathrm{P}, \mathbb{H})$ with finite trace, we assume that $\{W(t), t \in \mathbb{R}\}$ is a Q-Wiener process on $\mathbb{H}$.

Let $\mathbb{H}_0 = Q^{\frac{1}{2}}\mathbb{H}$ and $\mathcal{L}_2^0 = \mathcal{L}_2(\mathrm{P}, \mathbb{H}_0)$. It is easy to see that $\mathcal{L}_2^0$ is a separable Hilbert space with the norm

$$\|\Phi\|^2_{\mathcal{L}_2^0} = \|\Phi Q^{\frac{1}{2}}\|_2^2 = \mathrm{Trace}(\Phi Q \Phi^*).$$

Let $(\mathbb{X}, d)$ be a separable, complete metric space and $\mathrm{Pr}(\mathbb{X})$ be the set of Borel probability measures on $\mathbb{X}$.

We denote by $\mathcal{C}(\mathbb{R}, \mathbb{X})$ the class of all continuous functions from $\mathbb{R}$ to $\mathbb{X}$, and by $\mathcal{C}_b(\mathbb{X})$ the set of all continuous functions $\varphi : \mathbb{X} \to \mathbb{R}$ with $\|\varphi\|_\infty := \sup_{x\in\mathbb{X}} |\varphi(x)| < \infty$. For $\varphi \in \mathrm{C}_b(\mathbb{X})$, $\mu, \nu \in \mathrm{Pr}(\mathbb{X})$, we define

$$\|\varphi\|_L := \sup\left\{\frac{|\varphi(x) - \varphi(y)|}{d(x, y)};\ x, y \in \mathbb{X},\ x \neq y\right\},$$
$$\|\varphi\|_{BL} := \max\{\|\varphi\|_\infty, \|\varphi\|_L\},$$
$$d_{BL}(\mu, \nu) := \sup_{\|\varphi\|_{BL}\leq 1}\left|\int_{\mathbb{H}} \varphi d(\mu - \nu)\right|.$$

It is known that d_{BL} is a complete metric on $\mathrm{Pr}(\mathbb{X})$ which generates the weak topology [13]. For a random variable $x : (\Omega, \mathcal{F}, \mathrm{P}) \to \mathbb{X}$, we will denote by $\mathrm{P} \circ x^{-1}$ its distribution and by $\mathrm{E}(x)$ its expectation.

Definition 2.1 (*[14]*)**.** A stochastic process $x : \mathbb{R} \to \mathbb{H}$ is said to be stochastically continuous at the point $t_0 \in \mathbb{R}$ if for arbitrary $\epsilon > 0$ and $\delta > 0$, there exists $\rho > 0$ such that

$$\mathrm{P}(\|x(t) - x(t_0)\| > \varepsilon) < \delta, \quad \forall t \in [t_0 - \rho, t_0 + \rho].$$

Moreover, a stochastic process x is said to be stochastically continuous on the interval $I \subset \mathbb{R}$ if it is stochastically continuous at every point in I.

Definition 2.2. A stochastically continuous stochastic process $x : \mathbb{R} \to \mathcal{C}(\mathbb{R}, \mathbb{H})$ is said to be distributionally almost automorphic if every sequence $\{s_n'\}$ of real numbers has a subsequence $\{s_n\}$ such that for some stochastic process $\widetilde{x}$, the following:

$$\lim_{n\to\infty} d_{BL}(\mathrm{P} \circ [x(t + s_n)]^{-1}, \mathrm{P} \circ [\widetilde{x}(t)]^{-1}) = 0,$$

and

$$\lim_{n\to\infty} d_{BL}(\mathrm{P} \circ [\widetilde{x}(t - s_n)]^{-1}, \mathrm{P} \circ [x(t)]^{-1}) = 0,$$

hold for each $t \in \mathbb{R}$. That is, the continuous function $\mu : \mathbb{R} \to \mathrm{Pr}(\mathcal{C}(\mathbb{R}, \mathbb{H}))$ is almost automorphic.

The collection of all distributionally almost automorphic stochastic processes $x : \mathbb{R} \to \mathrm{Pr}(\mathcal{C}(\mathbb{R}, \mathbb{H}))$ is denoted by $AA(\mathbb{R}; \mathrm{Pr}(\mathcal{C}(\mathbb{R}, \mathbb{H})))$. It is clear that we have the following properties for $AA(\mathbb{R}; \mathrm{Pr}(\mathcal{C}(\mathbb{R}, \mathbb{H})))$.

Lemma 2.3. *If x, x_1 and x_2 are all distributionally almost automorphic stochastic processes, then:*

(1) *$x_1 + x_2$ is distributionally almost automorphic.*
(2) *λx is distributionally almost automorphic for every constant λ.*

Definition 2.4. A continuous function $f : \mathbb{R} \times \mathcal{C}(\mathbb{R}, \mathbb{H}) \to \mathcal{C}(\mathbb{R}, \mathbb{H})$, $(t, x) \to f(t, x)$ is said to be distributionally almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ if every sequence $\{s_n'\}$ of real numbers has a subsequence $\{s_n\}$ such that for some function $\widetilde{f}$, the following:

$$\lim_{n\to\infty} d_{BL}(\mathrm{P} \circ [f(t + s_n, x)]^{-1}, \mathrm{P} \circ [\widetilde{f}(t, x)]^{-1}) = 0,$$

and

$$\lim_{n\to\infty} d_{BL}(\mathrm{P}\circ[\widetilde{f}(t-s_n,x)]^{-1},\mathrm{P}\circ[f(t,x)]^{-1})=0,$$

hold for each $t\in\mathbb{R}$ and for each $x\in\mathcal{C}(\mathbb{R},\mathbb{H})$.

Theorem 2.5. *Let $f:\mathbb{R}\times\mathcal{C}(\mathbb{R},\mathbb{H})\to\mathcal{C}(\mathbb{R},\mathbb{H})$, $(t,x)\mapsto f(t,x)$ be distributionally almost automorphic in $t\in\mathbb{R}$ for every $x\in\mathcal{C}(\mathbb{R},\mathbb{H})$, and assume that f satisfies the Lipschitz condition in the following sense:*

$$d_{BL}(\mathrm{P}\circ[f(t,x]^{-1}),\mathrm{P}\circ[f(t,y)]^{-1})\le L\cdot d_{BL}(\mathrm{P}\circ[x(t)]^{-1},\mathrm{P}\circ[y(t)]^{-1}),$$

for all $x,y\in\mathcal{C}(\mathbb{R},\mathbb{H})$ and for each $t\in\mathbb{R}$, where $L>0$ is independent of t. Then for any distributionally almost automorphic process $x:\mathbb{R}\to\mathcal{C}(\mathbb{R},\mathbb{H})$, the stochastic process $F:\mathbb{R}\to\mathcal{C}(\mathbb{R},\mathbb{H})$ given by $F(t):=f(t,x(t))$ is distributionally almost automorphic.

Proof. Let $\{s_n'\}$ be a sequence of real numbers. By the almost automorphy of f and x, we can extract a subsequence $\{s_n\}$ of $\{s_n'\}$ such that for some function $\widetilde{f}$ and for each $t\in\mathbb{R}$ and $x\in\mathcal{C}(\mathbb{R},\mathbb{H})$,

$$\lim_{n\to\infty} d_{BL}(\mathrm{P}\circ[f(t+s_n,x)]^{-1},\mathrm{P}\circ[\widetilde{f}(t,x)]^{-1})=0,\tag{2.1}$$

and for some function $\widetilde{x}$ and for each $t\in\mathbb{R}$,

$$\lim_{n\to\infty} d_{BL}(\mathrm{P}\circ[x(t+s_n)]^{-1},\mathrm{P}\circ[\widetilde{x}(t)]^{-1})=0.\tag{2.2}$$

Let us consider the function $\widetilde{F}:\mathbb{R}\to\mathcal{C}(\mathbb{R},\mathbb{H})$ given by $\widetilde{F}(t):=\widetilde{f}(t,\widetilde{x}(t))$, $t\in\mathbb{R}$. Noting that

$$F(t+s_n)-\widetilde{F}(t)=f(t+s_n,x(t+s_n))-f(t+s_n,\widetilde{x}(t))+f(t+s_n,\widetilde{x}(t))-\widetilde{f}(t,\widetilde{x}(t)),$$

we have

$$\begin{aligned}
&d_{BL}(\mathrm{P}\circ[F(t+s_n)]^{-1},\mathrm{P}\circ[\widetilde{F}(t)]^{-1})\\
&\quad=\sup_{\|\varphi\|_{BL}\le1}\left|\int_{\mathbb{H}}\varphi d\mathrm{P}\circ[F(t+s_n)]^{-1}-\int_{\mathbb{H}}\varphi d\mathrm{P}\circ[\widetilde{F}(t)]^{-1}\right|\\
&\quad\le\sup_{\|\varphi\|_{BL}\le1}\int_{\Omega}|\varphi(F(t+s_n))-\varphi(\widetilde{F}(t))|d\mathrm{P}\\
&\quad\le\int_{\Omega}|F(t+s_n)-\widetilde{F}(t)|d\mathrm{P}\\
&\quad\le\int_{\Omega}|f(t+s_n,x(t+s_n))-f(t+s_n,\widetilde{x}(t))|d\mathrm{P}+\int_{\Omega}|f(t+s_n,\widetilde{x}(t))-\widetilde{f}(t,\widetilde{x}(t))|d\mathrm{P}\\
&\quad\le L\cdot d_{BL}(\mathrm{P}\circ[x(t+s_n)]^{-1},\mathrm{P}\circ[\widetilde{x}(t)]^{-1})+d_{BL}(\mathrm{P}\circ[f(t+s_n,\widetilde{x}(t))]^{-1},\mathrm{P}\circ[\widetilde{f}(t,\widetilde{x}(t))]^{-1}).
\end{aligned}$$

We deduce from (2.1) and (2.2) that

$$\lim_{n\to\infty} d_{BL}(\mathrm{P}\circ[F(t+s_n)]^{-1},\mathrm{P}\circ[\widetilde{F}(t)]^{-1})=0,\quad \text{for each } t\in\mathbb{R}.$$

Similarly we can prove that $\lim_{n\to\infty} d_{BL}(\mathrm{P}\circ[\widetilde{F}(t-s_n)]^{-1},\mathrm{P}\circ[F(t)]^{-1})=0$ for each $t\in\mathbb{R}$, which implies that $F(t)$ is distributionally almost automorphic. □

3. The main results

We consider the following affine stochastic differential equation:

$$dx(t)=[A(t)x(t)+f(t,x(t))]dt+g(t,x(t))dW(t),\quad t\in\mathbb{R},\tag{3.1}$$

where $A(t):D(A(t))\subset\mathbb{H}\to\mathbb{H}$ is a linear bounded operator on $\mathbb{H}$, both $f:\mathbb{R}\times\mathcal{C}(\mathbb{R},\mathbb{H})\to\mathcal{C}(\mathbb{R},\mathbb{H})$ and $g:\mathbb{R}\times\mathcal{C}(\mathbb{R},\mathbb{H})\to\mathcal{C}(\mathbb{R},\mathcal{L}_2^0)$ are nonlinear mappings, and $W(t)$ is a Q-Wiener process with values on $\mathbb{H}$.

Throughout this section, we also require the following assumptions.

(H1) Assumptions on the operator $A(t)$:
- (i) For all $t\in\mathbb{R}$, $A(t)$ generates a strongly continuous semigroup on $\mathbb{H}$.
- (ii) $A(\cdot)$ generates an evolution operator $U(t,s)$, $t\ge s$, on $\mathbb{H}$.
- (iii) $A(\cdot)$ is dissipative, that is, there exists a constant $K>0$ such that
$$\langle A(t)x,x\rangle\le -K\|x\|^2,$$
for every $t\in\mathbb{R}$ and $x\in\mathcal{C}(\mathbb{R},\mathbb{H})$.

(H2) Assumptions on f and g:
- (i) There exist positive numbers C_1,C_2 such that
$$\sup_{t\in\mathbb{R}}\|f(t,0)\|\le C_1,\qquad \sup_{t\in\mathbb{R}}\|g(t,0)\|^2_{\mathcal{L}_2^0}\le C_2.$$

(ii) There exists a constant $C_3 > 0$ such that
$$2\langle f(t,x) - f(t,y), x-y\rangle + \|g(t,x) - g(t,y)\|^2_{\mathcal{L}_2^0} \le C_3\|x-y\|^2,$$
for all stochastic process $x, y \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ and $t \in \mathbb{R}$.

Definition 3.1. An $\mathcal{F}_t$-progressively measurable process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution of (3.1) if it satisfies the following stochastic integral equation:

$$x(t) = U(t,a)x(a) + \int_a^t U(t,s)f(s,x(s))ds + \int_a^t U(t,s)g(s,x(s))dW(s), \tag{3.2}$$

for all $t \ge a$ and for each $a \in \mathbb{R}$.

A mild solution $x(t),\ t \in \mathbb{R}$ to (3.1) is said to be $\mathcal{L}_2$-bounded if

$$\sup_{t\in\mathbb{R}} \mathrm{E}(\|x(t)\|^2) < +\infty.$$

To prove an a priori estimate for the solution $x(t)$, we need the following assumption.

(H3) Let $A_n(t) = nA(t)(nI - A(t))^{-1}$ be the Yosida approximation of $A(t),\ t \in \mathbb{R}$, and let $U_n(t,s),\ t \ge s$, be the corresponding evolution operator. Then
$$\lim_{n\to\infty} U_n(t,s)x = U(t,s)x,$$
for all $x \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ and $t \ge s$.

Now we are going to prove an a priori estimate for the solution x.

Lemma 3.2. *Assume that* (H1)–(H3) *hold, and that ξ_a is $\mathcal{F}_a$-measurable with $\sup_{a\in\mathbb{R}} \mathrm{E}(\|\xi_a\|^2) = K_0 < \infty$. Let $x(t,a,\xi_a),\ t \ge a$, be the mild solution to* (3.1) *on $[a, +\infty)$. Then there exist constants $\omega > 0$ and $M > 0$ such that*

$$\mathrm{E}(\|x(t,a,\xi_a)\|^2) \le e^{-\omega t}\mathrm{E}(\|\xi_a\|^2) + M,$$

provided that $n > \frac{(2C_1^2 + C_3)K}{2K - C_3 - 2C_1^2}$.

Proof. Noting that

$$x(t) = \int_{-\infty}^t U(t,s)f(s,x(s))ds + \int_{-\infty}^t U(t,s)g(s,x(s))dW(s),$$

is well-defined and satisfies

$$x(t) = U(t,a)x(a) + \int_a^t U(t,s)f(s,x(s))ds + \int_a^t U(t,s)g(s,x(s))dW(s),$$

for all $t \ge a$ and each $a \in \mathbb{R}$, it is easy to see that x given by (3.2) is a mild solution to (3.1).

By (H1) and (H3), we have

$$\langle A_n(t)x, x\rangle = \left\langle \frac{nA(t)}{n - A(t)}x, x\right\rangle \le -\frac{nK}{n+K}\|x\|^2,$$

for all $x \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ and $t \in \mathbb{R}$. Let z_n be the mild solution of

$$z_n(t) = U_n(t,a)\xi_a + \int_a^t U_n(t,s)f(s,z_n(s))ds + \int_a^t U_n(t,s)g(s,z_n(s))dW(s).$$

By Itô's formula, we obtain

$$\mathrm{E}(\|z_n(t)\|^2) = \mathrm{E}(\|\xi_a\|^2) + 2\int_a^t \mathrm{E}\langle A_n(s)z_n(s) + f(s,z_n(s)), z_n(s)\rangle ds + \int_a^t \mathrm{E}\|g(s,z_n(s))\|^2_{\mathcal{L}_2^0} ds.$$

Then

$$\begin{aligned}
\frac{d}{dt}\mathrm{E}(\|z_n(t)\|^2) &= 2\mathrm{E}\langle A_n(t)z_n, z_n\rangle + 2\mathrm{E}\langle f(t,z_n(t)), z_n(t)\rangle + \mathrm{E}\|g(t,z_n(t))\|^2_{\mathbb{L}_2^0} \\
&\le -\frac{2nK}{n+K}\mathrm{E}(\|z_n(t)\|^2) + 2\mathrm{E}\langle f(t,z_n(t)) - f(t,0), z_n(t)\rangle + 2\mathrm{E}\langle f(t,0), z_n(t)\rangle \\
&\quad + \mathrm{E}(\|g(t,z_n) - g(t,0)\|^2_{\mathbb{L}_2^0}) + \mathrm{E}\|g(t,0)\|^2_{\mathbb{L}_2^0} \\
&\le -\left(\frac{2nK}{n+K} - C_3 - 2C_1^2\right)\mathrm{E}(\|z_n(t)\|^2) + C_2.
\end{aligned}$$

Integrating the above from a to t, we have

$$\mathrm{E}(\|z_n(t)\|^2) \le \mathrm{E}(\|\xi_a\|^2)e^{-\left(\frac{2nK}{n+K}-C_3-2C_1^2\right)(t-a)} + \frac{C_2(n+K)}{2nK-(C_3+2C_1^2)(n+K)}.$$

When $n > \frac{(2C_1^2+C_3)K}{2K-C_3-2C_1^2}$, the desired result follows with

$$\omega = \frac{2nK}{n+K} - C_3 - 2C_1^2$$

and

$$M = \frac{C_2(n+K)}{2nK-(C_3+2C_1^2)(n+K)}. \quad \square$$

We now prove the existence of $\mathcal{L}_2$-bounded solutions of (3.1) in all $\mathbb{R}$.

Proposition 3.3. *Under assumptions* (H1)–(H3), (3.1) *admits a unique $\mathcal{L}_2$-bounded continuous mild solution.*
Moreover the mapping $\hat{\mu} : \mathbb{R} \to \mathrm{Pr}(\mathcal{C}(\mathbb{R}, \mathbb{H}))$ defined by $\hat{\mu} := P \circ [x(t)]^{-1}$ is unique with the following properties:

(i)

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{H}} \|x\|^2 d\hat{\mu}(t)(x) < +\infty \quad (\mathcal{L}_2\text{-}boundedness), \tag{3.3}$$

(ii)

$$\mu(t, a, \hat{\mu}(a)) = \hat{\mu}(t) \quad \text{for all } t \ge a \text{ (the flow property)}. \tag{3.4}$$

For $\mu_0 \in \mathrm{Pr}(\mathcal{C}(\mathbb{R}, \mathbb{H}))$, $\mu(t, a, \mu_0)$ denotes the distribution of $x(t, a, \xi_a)$, where $\mu_0 = P \circ \xi_a^{-1}$.

Proof. For $t \ge -m \ge -n$, let $x_n := x(t, -n, 0)$ and $x_m := x(t, -m, 0)$; then it follows from Itô's formula that

$$\begin{aligned}\frac{d}{dt}\mathrm{E}(\|x_n(t) - x_m(t)\|^2) &= 2\mathrm{E}\langle A_n(t)[x_n(t) - x_m(t)], x_n(t) - x_m(t)\rangle \\ &\quad + 2\mathrm{E}\langle f(t, x_n(t)) - f(t, x_m(t)), x_n(t) - x_m(t)\rangle + \mathrm{E}\|g(t, x_n(t)) - g(t, x_m(t))\|^2_{\mathbb{L}_2^0} \\ &\le -\left(\frac{2nK}{n+K} - C_3\right)\mathrm{E}(\|x_n(t) - x_m(t)\|^2).\end{aligned}$$

By Lemma 3.2, we deduce that

$$\begin{aligned}\mathrm{E}(\|x_n(t) - x_m(t)\|^2) &\le \mathrm{E}(\|x(-m, -n, 0)\|^2)e^{-\left(\frac{2nK}{n+K}-C_3\right)(t+n)} \\ &\le K_0 e^{-\omega(t+n)} \to 0, \quad n \to \infty,\end{aligned}$$

where $\omega = -\frac{2nK}{n+K} + C_3$. Hence $x_n(t) \to x(t)$ in mean square. It is clear that $x(t)$ is $\mathcal{F}_t$-measurable and $\mathcal{L}_2$-bounded. From [8, Theorem 7.4(i)] it follows that $\{x(t)\}_{t\in\mathbb{R}}$ has a continuous version.

Now let $x(t)$ and $y(t)$ be two $\mathcal{L}_2$-bounded mild solutions to (3.1); then

$$\begin{aligned}\mathrm{E}(\|x(t) - y(t)\|^2) &= \mathrm{E}(\|x(t, -n, x(-n)) - x(t, -n, y(-n))\|^2) \\ &\le \mathrm{E}(\|x(-n) - y(-n)\|^2)e^{-\left(\frac{2nK}{n+K}-C_3-2C_1^2\right)(t+n)} \\ &\le K_0 e^{-\omega(t+n)} \to 0, \quad n \to \infty,\end{aligned}$$

where $\omega = -(\frac{2nK}{n+K} - C_3 - 2C_1^2)$ and thus $x(t) = y(t)$ for each $t \in \mathbb{R}$.

Since $x(t)$ is $\mathcal{F}_t$-adapted, it follows that $\mu = \mathrm{P} \circ x^{-1}$ satisfies the boundedness and the flow property. We now show the uniqueness.

Let μ_1 and μ_2 be two mappings which satisfy (3.3) and (3.4), and let ξ_1 and ξ_2 be random variables with distributions μ_1 and μ_2, respectively. Consider the solution $x(t, -n, \xi_1)$, $x(t, -n, \xi_2)$ to (3.1) over $[-n, +\infty)$. By the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
d_{BL}(\mu_1(t), \mu_2(t)) &= \sup_{\|\varphi\|_{BL}\le 1}\left|\int_{\mathbb{H}} \varphi d(\mu_1-\mu_2)\right| \\
&= \sup_{\|\varphi\|_{BL}\le 1}\left|\int_{\mathbb{H}} \varphi d\mathrm{P}\circ[x(t,-n,\xi_1)]^{-1} - \int_{\mathbb{H}} \varphi d\mathrm{P}\circ[x(t,-n,\xi_2)]^{-1}\right| \\
&= \sup_{\|\varphi\|_{BL}\le 1}\left|\int_{\Omega} \varphi(x(t,-n,\xi_1)) - \varphi(x(t,-n,\xi_2))d\mathrm{P}\right| \\
&\le \sup_{\|\varphi\|_{BL}\le 1}\int_{\Omega} |x(t,-n,\xi_1) - x(t,-n,\xi_2)|d\mathrm{P} \\
&\le \sqrt{K_0}e^{-\frac{\omega}{2}(t+n)} \to 0, \quad n\to\infty.
\end{aligned}$$

Thus $\mu_1(t) = \mu_2(t)$ for all $t\in\mathbb{R}$. □

We now consider the following stochastic differential equations:

$$dx_n(t) = [A_n(t)x_n(t) + f_n(t, x_n(t))]dt + g_n(t, x_n(t))dW(t), \quad n = 1, 2, \ldots, \tag{3.5}$$

where $A_n(t)$ are the Yosida approximation of $A(t)$, $t\in\mathbb{R}$. Let $U_n(t,s)$, $t\ge s$, be the corresponding evolution operator with respect to A_n. Assume that $f_n : \mathbb{R}\times\mathcal{C}(\mathbb{R},\mathbb{H}) \to \mathcal{C}(\mathbb{R},\mathbb{H})$, $(t,x)\to f_n(t,x)$ and $g_n : \mathbb{R}\times\mathcal{C}(\mathbb{R},\mathbb{H}) \to \mathcal{C}(\mathbb{R},\mathcal{L}_2^0)$, $(t,x)\to g_n(t,x)$ satisfy (H2), for each $n\in\mathbb{N}$.

For any fixed $a\in\mathbb{R}$, consider the corresponding integral equation

$$x_n(t) = U_n(t,a)\xi_a^n + \int_a^t U_n(t,s)f_n(s,x_n(s))ds + \int_a^t U_n(t,s)g_n(s,x_n(s))dW(s).$$

In the sequel, we need an additional assumption as follows.

(H4) For all $t\in\mathbb{R}$ and $x\in\mathcal{C}(\mathbb{R},\mathbb{H})$, it holds that

$$\lim_{n\to\infty} f_n(t,x) = f(t,x), \quad \text{in } \mathcal{C}(\mathbb{R},\mathbb{H}), \qquad \lim_{n\to\infty} g_n(t,x) = g(t,x), \quad \text{in } \mathcal{C}(\mathbb{R},\mathcal{L}_2^0).$$

The following result on the continuous dependence on initial data and coefficients will be useful:

Proposition 3.4 ([9]). *Suppose that* (H1)–(H4) *hold. Then we have:*

(i) *If* $\xi_a^n \xrightarrow{\mathbb{L}^2} \xi_a$, *then* $\sup_{t\in[a,+\infty)}\mathrm{E}(|x(t)-x_n(t)|^2)\to 0$.
(ii) *If* $\xi_a^n \xrightarrow{\mathbb{L}^p} \xi_a$, $p>2$, *then* $\mathrm{E}(\sup_{t\in[a,+\infty)}|x(t)-x_n(t)|^p)\to 0$.
(iii) *If* $d_{BL}(\mu_a^n,\mu_a)\to 0$ *in* $\Pr(\mathbb{H})$, *then* $d_{BL}(\mu(t,a,\mu_a^n),\mu(t,a,\mu_a))\to 0$ *in* $\Pr(\mathcal{C}(\mathbb{R},\mathbb{H}))$, *where* $\mu_a^n = P\circ[\xi_a^n]^{-1}$, $\mu_a = P\circ[\xi_a]^{-1}$.

Theorem 3.5. *Let A, f, g and $\{A_n, f_n, g_n\}_{n\in\mathbb{N}}$ satisfy* (H1)–(H4), *U and U_n be the evolution operators generated by A and A_n, and $x(t)$ and $x_n(t)$ be the unique $\mathcal{L}_2$-bounded mild solutions of* (3.1) *and* (3.5), *respectively. Assume in addition that:*

(i) *For every $t\in\mathbb{R}$, the family of distributions $\{\mathrm{P}\circ[x_n(t)]^{-1}\}_{n\in\mathbb{N}}$ is relatively compact.*
(ii) *For each $x\in\mathcal{C}(\mathbb{R},\mathbb{H})$, f and g are distributionally almost automorphic in $t\in\mathbb{R}$.*

Then the mapping

$$t\mapsto \mu_t = \mathrm{P}\circ[x(t)]^{-1}$$

is almost automorphic.

Proof. By the almost automorphy of f and g, we have

$$\lim_{n\to\infty} d_{BL}(\mathrm{P}\circ[f(t+s_n,x)]^{-1}, \mathrm{P}\circ[\widetilde{f}(t,x)]^{-1}) = 0,$$

and

$$\lim_{n\to\infty} d_{BL}(\mathrm{P}\circ[g(t+s_n,x)]^{-1}, \mathrm{P}\circ[\widetilde{g}(t,x)]^{-1}) = 0,$$

for each $t\in\mathbb{R}$ and $x\in\mathcal{C}(\mathbb{R},\mathbb{H})$.

Let $\widetilde{x}$ be the unique $\mathcal{L}_2$-bounded mild solution of the following stochastic differential equation:

$$dx(t) = [A(t)x(t) + \widetilde{f}(t,x(t))]dt + \widetilde{g}(t,x(t))dW(t), \quad t\in\mathbb{R}.$$

First, we prove that for every sequence $\{s_n'\} \subset \mathbb{N}$, there exists $\{s_n\} \subset \{s_n'\}$ such that for every $t \in \mathbb{R}$,

$$\lim_{n\to\infty} d_{BL}(\mathrm{P}\circ[x(t+s_n)]^{-1}, \mathrm{P}\circ[\widetilde{x}(t)]^{-1}) = 0. \tag{3.6}$$

Write $\widetilde{\mu}_t = \mathrm{P}\circ[\widetilde{x}(t)]^{-1}$. For simplicity, we assume that $s_n' = n$. For every $r \geq 1$ we have

$$\begin{aligned} x_n(t) &:= x(t+n) \\ &= U(t+n, -r+n)x(-r+n) + \int_{-r+n}^{t+n} U(t+n,\sigma)f(\sigma, x(\sigma))d\sigma \\ &\quad + \int_{-r+n}^{t+n} U(t+n,\sigma+n)g(\sigma, x(\sigma))dW(\sigma). \end{aligned}$$

Put $\widetilde{W}(s) = W(s+n) - W(n)$ for each s. Note that $\widetilde{W}$ is also a Q-Wiener process and has the same distribution as W. Now, let us make an appropriate change of variable to get

$$\begin{aligned} x_n(t) &= U(t+n, -r+n)x(-r+n) + \int_{-r}^{t} U(t+n,s)f(s+n, x(s+n))ds \\ &\quad + \int_{-r}^{t} U(t, s+n)g(s+n, x(s+n))d\widetilde{W}(s) \\ &= U_n(t,-r)x_n(-r) + \int_{-r}^{t} U_n(t,s)f_n(s, x_n(s))ds + \int_{-r}^{t} U_n(t,s)g_n(s, x_n(s))d\widetilde{W}(s). \end{aligned}$$

Since $\{\mathrm{P}\circ x_n^{-1}(-r)\}_{n\in\mathbb{N}}$ is relatively compact for any fixed $n \in \mathbb{N}$, there are $\{s_n^r\} \subset n$ and $\{s_n^{r+1}\} \subset \{s_n^r\}$ such that $x(-r+s_n^r)$ and $x(-r+s_n^{r+1})$ are weak limits of the sequence $x(-r+n)$, and we deduce that

$$P\circ[x(-r+s_n^r)]^{-1} = P\circ[x(-r+s_n^{r+1})]^{-1}, \quad t \geq -r.$$

Then we can define $\lim_{n\to\infty} P\circ[x(-r+s_n^r)]^{-1} = \mu_r \in \mathrm{Pr}(\mathcal{C}(\mathbb{R},\mathbb{H}))$. From [8], there exists a random variable $\xi_r : \Omega \to \mathcal{C}(\mathbb{R},\mathbb{H})$ such that $\mathrm{P}\circ\xi_r^{-1} = \mu_r$.

For $t \geq -r$, we consider the following processes:

$$\begin{aligned} x(t+s_n^r) &= U_{s_n^r}(t,-r)x(-r+s_n^r) + \int_{-r}^{t} U_{s_n^r}(t,s)f(s+s_n^r, x(s+s_n^r))ds \\ &\quad + \int_{-r}^{t} U_{s_n^r}(t,s)g(s+s_n^r, x(s+s_n^r))d\widetilde{W}(s), \end{aligned}$$

$$y^r(t) = U(t,-r)\xi_r + \int_{-r}^{t} U(t,s)\widetilde{f}(s,\widetilde{x}(s))ds + \int_{-r}^{t} U(t,s)\widetilde{g}(s,\widetilde{x}(s))dW(s)$$

$$y_n^r(t) = U(t,-r)x(-r) + \int_{-r}^{t} U(t,s)\widetilde{f}(s,\widetilde{x}(s))ds + \int_{-r}^{t} U(t,s)\widetilde{g}(s,\widetilde{x}(s))dW(s).$$

By Proposition 3.4, we have

$$d_{BL}(\mathrm{P}\circ[y_n^r(t)]^{-1}, \mathrm{P}\circ[y^r(t)]^{-1}) \to 0, \quad n\to\infty, \tag{3.7}$$

and

$$d_{BL}(\mathrm{P}\circ[x(t+s_n^r)]^{-1}, \mathrm{P}\circ[y_n^r(t)]^{-1}) \to 0, \quad n\to\infty. \tag{3.8}$$

Combining (3.7) with (3.8), we have

$$d_{BL}(\mathrm{P}\circ[x(t+s_n^r)]^{-1}, \mathrm{P}\circ[y^r(t)]^{-1}) \to 0, \quad n\to\infty.$$

Put $s_n = s_n^n$. Then we have

$$d_{BL}(\mathrm{P}\circ[x(t+s_n)]^{-1}, \mathrm{P}\circ[y^r(t)]^{-1}) \to 0, \quad n\to\infty. \tag{3.9}$$

For every $t \in [-r, +\infty)$, we see that both $y^r(t)$ and $y^{r+1}(t)$ are weakly convergent sequences of $x(t+s_n)$. Hence we have for $t \geq -r$,

$$\mathrm{P}\circ[y^r(t)]^{-1} = \mathrm{P}\circ[y^{r+1}(t)]^{-1}, \quad r\to+\infty.$$

Therefore, we can define $\widetilde{\mu}(t) := \lim_{r\to+\infty} \mathrm{P}\circ[y^r(t)]^{-1}$ for all $t \geq -r$.

Equivalently,

$$d_{BL}(\mathrm{P} \circ [x(t+s_n)]^{-1}, \widetilde{\mu}(t)) \to 0, \quad n \to \infty,$$

for each $t \in \mathbb{R}$. By Fatou's Lemma, we obtain that

$$\int_{\mathbb{H}} \|x\|^2 d\widetilde{\mu}(t)(x) \le K_1 < \infty.$$

For any fixed $r \in \mathbb{R}$, when $t \ge s \ge -r$, we have

$$y^r(t) = U(t,s)y^r(s) + \int_s^t U(t,\sigma)\widetilde{f}(\sigma, y^r(\sigma))d\sigma + \int_s^t U(t,s)\widetilde{g}(s, y^r(\sigma))dW(\sigma),$$

i.e.,

$$\widetilde{\mu}_r(t) = \mu(t, s, \widetilde{\mu}_r(s)).$$

By Proposition 3.3, this characterizes $\widetilde{\mu}$ uniquely, and (3.6) is shown.

Similarly, we can obtain

$$\lim_{n\to\infty} d_{BL}(\mathrm{P} \circ [\widetilde{x}(t-s_n)]^{-1}, \mathrm{P} \circ [x(t)]^{-1}) = 0,$$

for each $t \in \mathbb{R}$. The proof is complete. □

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